(Continuous) Fourier Series Examples

Standard types of question:

① Find the Fourier coefficients of $f(t)$

i.e. compute $\mathcal{F}\{f(t)\}$

② Write the Fourier series for $f(t)$

i.e. compute $c_n = \mathcal{F}_n\{f(t)\}$

and write $f(t) = \sum_{-\infty}^{\infty} c_n e^{int}$

Other possibilities:

③ Find the amplitude of the frequency n component of $f(t)$

④ Find the phase of the frequency n component of $f(t)$

⑤ Convert between Sine/Cosine and $\mathbb{C}$-Exponential forms of Fourier series

EX: Compute the coefficients $\mathcal{F}\{1\}$.

$$c_n = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} 1\cdot e^{-int}\,dt = \frac{1}{2\pi}\frac{1}{-in}e^{-int}\Big|_{-\pi}^{\pi}$$

$$= \tfrac{1}{2\pi}\frac{1}{-in}\left(e^{-in\pi} - e^{in\pi}\right) \qquad e^{\pm in\pi} = (-1)^n$$

$$= \frac{1}{2\pi}\tfrac{i}{n}\left((-1)^n - (-1)^n\right)$$

$$= 0 \qquad (\text{for } n \neq 0)$$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dt = 1$$

EX: Compute the coefficients $\mathcal{F}\{5\sin 3t\}$.

Plan: Convert from Sine/Cosine series

$$5\sin(3t) = a_0/2 + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt)$$

where all $a_n = 0$

and $b_3 = 5$ other $b_n = 0$

↳ Exponential coefficients:

$$c_n = \frac{a_n - ib_n}{2}$$

$$\text{So } \begin{cases} c_3 = -\tfrac{5}{2}i \\ c_{-3} = \tfrac{5}{2}i \end{cases}, \quad \text{other } c_n = 0.$$

EX: Compute $\mathcal{F}\{5\sin 3t \cos 3t\}$

Plan: Convert from Sine/Cosine series

$$5\sin 3t\cos 3t = 5/2 \cdot \sin 6t$$

(Recall: $2\sin A\cos A = \sin 2A$)

$$= a_0/2 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

where all $a_n = 0$

and $b_6 = 5/2$, other $b_n = 0$

$$\begin{cases} c_6 = -5/4\, i \\ c_{-6} = 5/4\, i \end{cases} \quad \text{other } c_n = 0$$

EX: Compute $\mathcal{F}\{2\cos 3t + 7\sin 3t\}$

Plan: Convert from Sine/Cosine series

$$2\cos 3t + 7\sin 3t = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

where $a_3 = 2$ and $b_3 = 7$

other $a_n, b_n = 0$.

$$\begin{cases} c_6 = \dfrac{2-7i}{2} = 1 - 7/2\, i \\ c_{-6} = 1 + 7/2\, i \end{cases}$$

other $c_n = 0$.

EX: Compute $\mathcal{F}\{t\}$

Plan: Use the definition — Integrate!

$$c_n = 1/2\pi \int_{-\pi}^{\pi} t\, e^{-int}\, dt$$

Int. by Parts.

$$= 1/2\pi \left(\frac{1}{-in} t e^{-int} - \frac{1}{(in)^2} e^{-int} \right) \Big|_{-\pi}^{\pi}$$

$\frac{1}{i} = -i$

$$= 1/2\pi \left(i/n\, \pi e^{-in\pi} + 1/n^2\, e^{-in\pi} - i/n(-\pi) e^{in\pi} - 1/n^2\, e^{in\pi} \right)$$

$e^{\pm in\pi} = (-1)^n$

$$= 1/2\pi \left(\frac{2\pi}{n} (-1)^n i \right)$$

$$= (-1)^n\, 1/n\, i \qquad \text{for } n \neq 0.$$

$$c_0 = 1/2\pi \int_{-\pi}^{\pi} t\, dt = 0.$$

EX: Compute $\mathcal{F}\{t^2\}$

Plan: Use formula $\mathcal{F}_n\{\int f(t)\} = -i/n\, \mathcal{F}_n\{f(t)\}$

→ since $\mathcal{F}_0\{t\} = 0$.

$$t^2 = 2\int t\, dt$$

$$c_n = \mathcal{F}_n\{t^2\} = \mathcal{F}_n\{2\int t\, dt\} = 2(-i/n)\, \mathcal{F}_n\{t\}$$

$$= 2(-i/n)(-1)^n\, 1/n\, i$$

$$= (-1)^n\, 2/n^2 \qquad \left(-(-1)^n i^2 = (-1)^n\right)$$

$$c_0 = \mathcal{F}_0\{t^2\} = 1/2\pi \int_{-\pi}^{\pi} t^2\, dt = 1/3\, \pi^2$$

EX: Compute $\mathcal{F}\{\delta(t)\}$ ← Impulse at $t=0$.

Plan: Integrate!

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(t)\cdot e^{-int}\,dt$$
$$= \frac{1}{2\pi} e^{-in(0)}$$
$$= \frac{1}{2\pi}$$

(**Recall:** $\int \delta(t-c)\, f(t)\, dt = f(c)$)

EX: Compute $\mathcal{F}\{u_0(t)\}$ ← Step function $\begin{cases} 0 & t<0 \\ 1 & 0\le t\end{cases}$

Plan: Integrate!

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_0(t)\cdot e^{-int}\,dt$$
$$= \frac{1}{2\pi}\left(\int_{-\pi}^{0} 0\cdot e^{-int}\,dt + \int_0^{\pi} 1\cdot e^{-int}\,dt\right)$$
$$= \frac{1}{2\pi}\int_0^{\pi} e^{-int}\,dt$$
$$= \frac{1}{2\pi}\,\frac{1}{-in}\, e^{-int}\Big|_0^{\pi}$$
$$= \frac{1}{2\pi}\,\frac{i}{n}\left(e^{-in\pi} - e^{0}\right)$$
$$= \frac{1}{2\pi}\,\frac{i}{n}\left((-1)^n - 1\right)$$
$$= \frac{1}{2\pi n}\left((-1)^n - 1\right) i \qquad (n\neq 0)$$
$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_0(t)\,dt = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$$

EX: Compute $\mathcal{F}\{u_0(t)\}$

Plan: Use formula $\mathcal{F}\{\int f\} = \frac{i}{n}\cdot\left((-1)^n c_0 - c_n\right)$

$$u_0(t) = \int \delta(t)\,dt$$
$$\mathcal{F}_n\{u_0\} = \frac{i}{n}\left((-1)^n \mathcal{F}_0\{\delta\} - \mathcal{F}_n\{\delta\}\right)$$
$$= \frac{i}{n}\left((-1)^n \frac{1}{2\pi} - \frac{1}{2\pi}\right)$$
$$= \frac{1}{2\pi n}\left((-1)^n - 1\right) i$$

$$\mathcal{F}_0\{u_0\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_0(t)\,dt = \frac{1}{2}$$

Must still calculate coefficient 0 by hand with integral...

EX: Compute $\mathcal{F}\{u_c(t)\}$

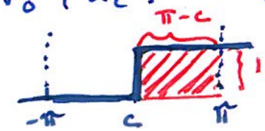
First compute $\mathcal{F}_n\{\delta(t-c)\}$

$$\mathcal{F}_n\{\delta(t-c)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(t-c)\, e^{-int}\,dt = \frac{e^{-inc}}{2\pi}$$
$$\mathcal{F}_0\{\delta(t-c)\} = \frac{1}{2\pi}$$

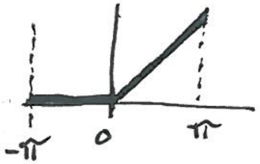
Now get $\mathcal{F}_n\{u_c\} = \mathcal{F}_n\{\int \delta(t-c)\}$

$$\mathcal{F}_n\{u_c\} = \frac{i}{n}\left((-1)^n\frac{1}{2\pi} - \frac{1}{2\pi}e^{-inc}\right)$$

$$\mathcal{F}_0\{u_c\} = \frac{\pi - c}{2\pi}$$ ← average value of $u_c(t)$ on $(-\pi,\pi)$.



EX: Compute $\mathcal{F}\{u_0(t)\cdot t\}$



Method 1: Integration!

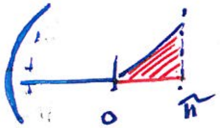
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_0(t)\cdot t\cdot e^{-int}\,dt$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} t\cdot e^{-int}\,dt \qquad \text{Int. by parts}$$

$$= \frac{1}{2\pi}\left(\frac{i}{n}te^{-int} + \frac{1}{n^2}e^{-int}\right)\Bigg|_0^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{i}{n}\pi e^{-in\pi} + \frac{1}{n^2}e^{-in\pi} - \cancel{\frac{i}{n}0e^{0}} - \frac{1}{n^2}e^{0}\right)$$

$$= \frac{1}{2\pi n}\left((-1)^n\pi i + \frac{1}{n}\left((-1)^n - 1\right)\right)$$

$$c_0 = \frac{1}{2\pi}\Bigg($$  $$\Bigg) = \frac{1}{2\pi}\cdot\frac{1}{2}\pi^2$$

$$= \frac{1}{4}\pi$$

Method 2: Manipulations

$$u_0(t)\cdot t = \int u_0(t)\,dt$$

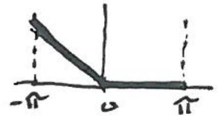
$$\mathcal{F}_n\{u_0(t)\cdot t\} = \frac{i}{n}\left((-1)^n\mathcal{F}_0\{u_0\} - \mathcal{F}_n\{u_0\}\right)$$

$$= \frac{i}{n}\left((-1)^n\cdot\frac{1}{2} - \frac{1}{2\pi n}\left((-1)^n - 1\right)i\right)$$

$$= \frac{1}{2n}\left((-1)^n i + \frac{1}{\pi n}\left((-1)^n - 1\right)\right)$$

$$\mathcal{F}_0\{u_0(t)\cdot t\} = \cdots = \frac{1}{4}\pi.$$

EX: Compute $\mathcal{F}\{f(t)\}$



$$f(t) = \begin{cases} -t & t<0 \\ 0 & 0\le t \end{cases}$$

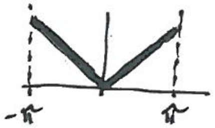
Note that $f(t) = u_0(-t)\cdot(-t)$ is the reflection across y-axis of previous problem.

↳ this takes conjugate of c_n

$$c_n = \frac{1}{2\pi n}\left(-(-1)^n\pi i + \frac{1}{n}\left((-1)^n - 1\right)\right)$$

$$c_0 = \frac{1}{4}\pi$$

EX: Compute $\mathcal{F}\{f(t)\}$



$$f(t) = |t|$$

This is the sum of the previous two

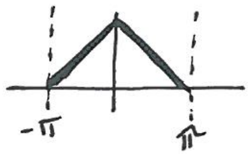
↳ add Fourier coefficients.

$$c_n = \frac{1}{2\pi n}\left(\cancel{(-1)^n\pi i} + \frac{1}{n}\left((-1)^n - 1\right)\right) + \frac{1}{2\pi n}\left(\cancel{-(-1)^n\pi i} + \frac{1}{n}\left((-1)^n - 1\right)\right)$$

$$= \frac{1}{\pi n^2}\left((-1)^n - 1\right)$$

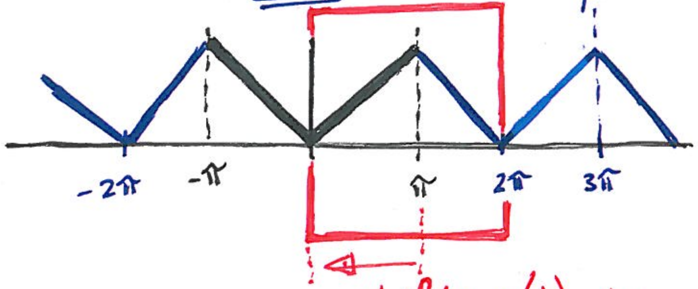
$$c_0 = \frac{1}{2}\pi$$

EX: Compute $\mathcal{F}\{f(t)\}$



$f(t) = 1 - |t|$

This is a shift of the previous problem.



shift: $g(t)$ to $g(t-\pi)$

$$\text{So} \quad c_n = \mathcal{F}_n\{g(t-\pi)\}$$

$$= e^{-in\pi} \cdot \mathcal{F}_n\{|t|\}$$

$$= (-1)^n \cdot \left(\frac{1}{\pi n^2} \left((-1)^n - 1 \right) \right)$$

$$= \frac{1}{\pi n^2} \left((-1)^{2n} - (-1)^n \right)$$

$$= \frac{1}{\pi n^2} \left(1 - (-1)^n \right)$$

$$c_0 = \mathcal{F}_0\{g(t-\pi)\} = \mathcal{F}_0\{|t|\}$$

$$= \tfrac{1}{2}\pi$$

Remember: $|t|$ was supposed to be extended periodically outside $(-\pi, \pi)$